

A Simple Lambert Algorithm

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A classic result for the two-point boundary value problem in the framework of Keplerian motion allows the derivation of a novel parametrization of orbits passing through two arbitrary points in space. In particular, it is shown that these orbits can be unambiguously identified in terms of their eccentricity vector component in the direction perpendicular to the chord connecting the two points. The parametrization, in terms of transverse eccentricity component, lends itself to an efficient and intuitive solution algorithm for the classical Lambert problem, that is, the determination of the orbit that connects two points in space in a prescribed time. Although, from the computational point of view, the resulting numerical procedure does not provide advantages over the elegant Battin's method, its derivation is considerably less demanding from the mathematical standpoint and physically more intuitive.

Nomenclature

a	=	semimajor axis
c	=	chord
e, \mathbf{e}	=	eccentricity and eccentricity vector
e_F	=	eccentricity component along the chord
e_T	=	transverse eccentricity component
$\hat{\mathbf{i}}_c, \hat{\mathbf{i}}_p, \hat{\mathbf{i}}_h$	=	chord, transverse, and normal unit vectors
p	=	semilatus rectum or parameter
r, \mathbf{r}	=	radius and radius vector
t_{12}	=	transfer time
x	=	$\xi(e_T)$, transformed parameter
y	=	$\log(\tau_{12})$, transformed transfer time
$\Delta\theta$	=	transfer angle
$\Delta\omega$	=	rotation of the periapsis
$\varepsilon_x, \varepsilon_y$	=	tolerances
θ	=	anomaly, counted from \mathbf{r}_1
μ	=	gravitational parameter
ν	=	true anomaly, counted from the periapsis
ρ	=	r_2/r_1 , ratio of radii
τ_{12}	=	nondimensional transfer time
ω	=	argument of the perigee

Subscripts and Superscripts

F	=	relative to the fundamental ellipse
T	=	transverse
S	=	specified
1, 2	=	relative to P_1 or P_2

I. Introduction

THE two-point boundary value problem for Keplerian motion, also known as Lambert's problem, is a classical one in astrodynamics. Following Battin's formulation [1], the problem can be stated as the "determination of an orbit, having a specified transfer time and connecting two position vectors." In this paper, a novel solution method to Lambert's problem is discussed, based on an original parametrization of admissible orbits passing through two

given points. The situation is depicted in Fig. 1, where P_1 and P_2 are the initial and final positions of the orbiting body, respectively. The angle $\Delta\theta$ between the two positions vectors \mathbf{r}_1 and \mathbf{r}_2 based on the occupied focus F of the system is the transfer angle.

Indicating the specified transfer time between P_1 and P_2 as t_{12}^S , Lambert problem is defined by four parameters, namely, $r_1 = \|\mathbf{r}_1\|$, $r_2 = \|\mathbf{r}_2\|$, $\Delta\theta$, and t_{12}^S . The scope is to determine the orbit elements of an orbit through P_1 and P_2 (that is, eccentricity e , semimajor axis a , or, equivalently, parameter p , and argument of the perigee ω) such that the time-of-flight between P_1 and P_2 matches the desired value.

The problem stems directly from Lambert's theorem [2], which states that the transfer time between any two points in the framework of the two-body problem depends on three parameters only, namely, the chord length between the points c , the sum of the norm of the radius vectors $r_1 + r_2$, and the semimajor axis of the considered orbit. This means that, when the geometry of the radius vectors is fixed, there is only one free parameter left that wholly defines the transfer time between P_1 and P_2 . As a consequence, once a suitable parametrization of the orbits passing through P_1 and P_2 is obtained, it is possible to rephrase Lambert's problem in terms of the evaluation of the parameter value, such that the corresponding orbit is characterized by a transfer time that matches exactly the prescribed one.

The relevance of Lambert's problem to astrodynamics is somehow self-evident, as an efficient solution algorithm can be extremely useful in almost any part of space mission design and operation, from orbit determination [2] to trajectory optimization [3]. In the absence of an analytical solution, all the approaches discussed in the literature are based on a numerical procedure where the value of the free parameter is searched iteratively. After the early works of Gauss, between the end of the 18th and the beginning of the 19th century [4], the properties of the conic sections passing through two given points were searched extensively and several algorithms were devised for tackling Lambert's problem [5–9]. It is also possible to state the problem in terms of universal variables, so that a uniform parametrization is obtained for all type of transfers along arcs of elliptical, parabolic, or hyperbolic orbits [10].

Numerical convergence of different methods and their overall computational cost depend on both the chosen parametrization and the numerical technique used for solving the resulting equation. Singularities exist, which prevent some of the algorithms from converging for particular cases or make convergence extremely slow (e.g., Gauss's method fails for 180 deg transfer angles [8]). Battin and Vaughan emphasized that their discovery of an algorithm not affected by this flaw was their most significant achievement [8].

Battin's method relies at least partially on geometrical properties of the family of conic curves passing through P_1 and P_2 , which will be termed in the sequel, the set of admissible orbits. These features are described in detail in his seminal works [6–8] and are recalled in

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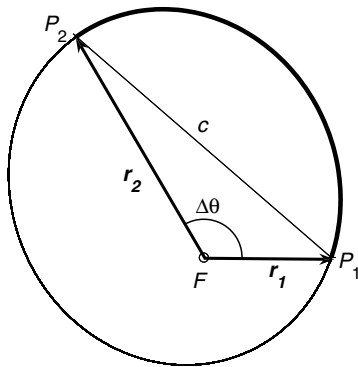


Fig. 1 Geometry of the Lambert problem.

his renowned book [1], which provides the astrodynamics community with one of the most widely used numerical procedures to efficiently solve Lambert's problem. His method is based on a corollary to Lambert's theorem [6], which states that the mean radius r_0 depends on the same set of variables as the transfer time, where the mean radius is defined as the radius of the point along the orbit where the velocity vector is parallel to the chord connecting P_1 with P_2 (the normal point according to Levine's definition [11]).

Given the invariance property stated by Lambert's theorem, it is possible to demonstrate that the occupied and the vacant foci of orbits passing through P_1 and P_2 must be moved along two confocal ellipses, with P_1 and P_2 as the foci, to maintain the same transfer time [6]. Among all the admissible ellipses characterized by the same transfer time, the symmetric one is of interest. In such a case, the normal radius is perpendicular to the chord and it lies along the line of the apsides, being equal to either the orbit perigee or apogee, depending on the relative position of the occupied and vacant foci with respect to the chord. The transfer time between P_1 and P_2 can thus be determined by solving Kepler's equation only once, along half of the transfer, the true anomaly of P_1 and P_2 being equal in magnitude and opposite in sign, hence their eccentric anomalies E . The set of symmetric orbits can be parametrized in terms of an auxiliary variable x equal to the tangent of half of the eccentric anomaly E . Lambert's problem is thus solved by evaluating the value of x corresponding to the symmetric orbit with the prescribed transfer time. The final orbit is determined by back-transforming the problem geometry, shifting the occupied focus to its original position. The extension of the approach to hyperbolic trajectories is dealt with in Chapter 7 of Battin's book [1].

The need for massive computations in the framework of optimization problems puts significant stress on numerical performance and, in this respect, the success of Battin's method is also a consequence of its computational efficiency and well-posedness of the resulting equation. But if on one side the derivation of the equation is relatively simple, its solution is far from trivial, requiring relatively cumbersome transformations and the evaluation of a hypergeometric function in terms of its (truncated) continued fraction expansion [8]. Moreover, care is needed when dealing with very short arcs to provide a satisfactory first guess for the algorithm [8].

The novel parametrization introduced in this paper allows for a numerical solution of Lambert's problem almost as efficient as Battin's method, using a simple Newton–Raphson iterative scheme. If, on one side, no significant advantages over Battin's method resulted in terms of accuracy, convergence speed, and overall numerical efficiency, its simplicity makes it an appealing alternative to the present-day classic Battin's approach. This parametrization exploits a well-known property of the eccentricity vector e , which has a constant component in the direction of the chord $r_1 - r_2$. In [1] (p. 256), this property is attributed to E. Bender of the Jet Propulsion Laboratory. The fundamental ellipse, that is, the orbit through P_1 and P_2 with minimum eccentricity, can be easily determined on geometrical grounds. Given the fundamental ellipse, it is possible to obtain a parametrization of admissible orbits in terms of the transverse eccentricity component e_T , that is, the component of the

eccentricity vector in the direction perpendicular to the chord. It will be shown that the transfer time t_{12} between P_1 and P_2 is a monotonic function of e_T , so that it is possible to use e_T as the unknown for solving Lambert's problem by means of the equation $t_{12}(e_T) = t_{12}^S$. Like Battin's method, the resulting algorithm is not singular for $\Delta\theta = \pi$.

The value of the transverse eccentricity component must satisfy some constraints, that is, its value is limited within a (possibly finite) interval. For this reason, a coordinate transformation is devised, such that the numerical iterative scheme is not allowed to violate the admissible limits of variation for the unknown e_T . In the present paper, only direct transfer arcs between P_1 and P_2 are considered. Multiple-revolution transfers are the subject of ongoing research.

In what follows, after a brief review of the results reported by Battin on the fundamental ellipse, the parametrization of orbits passing through P_1 and P_2 in terms of transverse eccentricity component e_T will be derived, together with the transfer time. In this framework, some minor variations to Battin's notation will be introduced to provide a uniform parametrization that embraces all possible values of r_1 , r_2 , and $\Delta\theta$. Lambert's problem is here parametrized in nondimensional terms, in such a way that the (nondimensional) transfer time τ_{12} becomes a function of two parameters only, namely, the transfer angle $\Delta\theta$ and the ratio of the radii r_2/r_1 . In Sec. IV, the coordinate transformation used for providing a fail-proof implementation of the Newton–Raphson iterative scheme is described and results are derived for a set of sample problems. Comparisons in terms of computational efficiency with existing methods will be discussed. The Conclusions section ends the paper.

II. Fundamental Ellipse

The orbit equation is expressed in vector form as [1]

$$e \cdot r = p - r \quad (1)$$

which is represented in a set of polar coordinates (r, θ) by the equation

$$r(\theta) = \frac{p}{1 + e \cos(\theta - \omega)} \quad (2)$$

where the angular displacement θ is positive for counterclockwise rotations, and the argument of the periapsis ω is the angle between the reference direction and the eccentricity vector e . Without loss of generality, it is possible to assume the direction of $\hat{r}_1 = r_1/r_1$ as the reference axis of the polar coordinates.

As shown in [1], by taking the difference of the orbit equation in vector form written for r_2 and r_1 , one gets

$$e \cdot (r_2 - r_1) = r_1 - r_2$$

Dividing both terms of the latter equation by the length of the chord $c = \|r_2 - r_1\|$, one gets

$$e \cdot \hat{c} = -\frac{r_2 - r_1}{c} \quad (3)$$

which clearly shows that the eccentricity vectors of all the orbits through P_1 and P_2 have the same projection $e_F = |r_2 - r_1|/c$ along the chord direction. This means that 1) all the eccentricity vectors of admissible orbits terminate on a straight line perpendicular to the chord c at distance e_F from the occupied focus F , and 2) the minimum possible eccentricity is e_F , when the component in the transverse direction is zero.

The minimum eccentricity orbit is referred to as the fundamental ellipse. Its eccentricity vector is given by $e_F = [(r_1 - r_2)/c]\hat{c}$, whereas its semimajor axis is expressed on the basis of simple symmetry considerations as [1]

$$a_F = (r_1 + r_2)/2$$

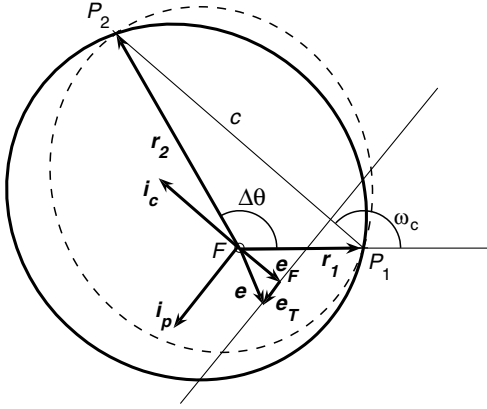


Fig. 2 Fundamental ellipse (continuous line) and a generic admissible orbit (dashed line).

The orbit parameter p_F is determined accordingly as $p_F = a_F(1 - e_F^2)$.

The eccentricity vector of a generic orbit passing through P_1 and P_2 can be represented as the sum of the constant eccentricity component of the fundamental ellipse e_F , parallel to the chord, plus a transverse component, in the direction perpendicular to both the chord and e_F (Fig. 2),

$$\mathbf{e} = \mathbf{e}_F + \tan(\Delta\omega)(\hat{\mathbf{i}}_h \times \mathbf{e}_F) \quad (4)$$

where $\hat{\mathbf{i}}_h$ is the unit vector parallel to the angular momentum vector (that is, perpendicular to the orbit plane), and $\Delta\omega$ is the rotation of the periastris with respect to the eccentricity vector of the fundamental ellipse (ω in Battin's notation). It is possible to show that the semilatus rectum p of the corresponding orbit is given by [1]

$$p(\Delta\omega) = p_F + \sigma(r_2 - r_1)e_F \tan(\Delta\omega) \frac{r_1 r_2}{c} \sin(\Delta\theta) \quad (5)$$

where the sign function $\sigma(\cdot)$ is defined as

$$\sigma(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Note that the original formulation of Eq. (5) (i.e., Eq. 6.59 on p. 262 of [1]) does not feature the sign function and it is thus valid only if $r_1 < r_2$, when the minimum eccentricity vector \mathbf{e}_F is in the direction opposite to the chord unit vector $\hat{\mathbf{i}}_c$. In such a case, it is $\mathbf{e} \times \hat{\mathbf{i}}_c = e_F \tan \Delta\omega \hat{\mathbf{i}}_h$. On the converse, when $r_1 > r_2$, \mathbf{e}_F is parallel to the chord unit vector $\hat{\mathbf{i}}_c$ and $\mathbf{e} \times \hat{\mathbf{i}}_c = -e_F \tan \Delta\omega \hat{\mathbf{i}}_h$. As a consequence, when $r_1 > r_2$, the second term on the right-hand side of Eq. (5) must feature a minus sign.

Although more general than the original formulation, Eq. (5) is not valid when $r_1 = r_2$ and $e_F = 0$. In this latter case, the fundamental ellipse becomes a circle and the angle $\Delta\omega$ is no longer defined, if referred to \mathbf{e}_F ; $\Delta\omega$ becomes equal to either $\pi/2$ or $-\pi/2$ if counted with respect to $\hat{\mathbf{i}}_c$.

III. Orbit Parametrization in Terms of Transverse Eccentricity

A. Preliminary Considerations

Given the results mentioned in the previous paragraph, it is possible to revisit them to provide a uniform parametrization of orbits connecting P_1 and P_2 and the corresponding transfer time t_{12} as a function of the (variable) transverse eccentricity component e_T . First of all, a sign is attributed to both e_F and e_T . In particular, it is

$$e_F = \frac{r_1 - r_2}{c} \quad (6)$$

such that $\mathbf{e}_F = e_F \hat{\mathbf{i}}_c$. Letting $\hat{\mathbf{i}}_p = \hat{\mathbf{i}}_h \times \hat{\mathbf{i}}_c$ be the unit vector lying in the orbit plane and perpendicular to the chord direction, the

eccentricity vector of a generic orbit passing through P_1 and P_2 is now written as

$$\mathbf{e} = e_F \hat{\mathbf{i}}_c + e_T \hat{\mathbf{i}}_p \quad (7)$$

The eccentricity of the orbit becomes

$$e(e_T) = \sqrt{e_F^2 + e_T^2} \quad (8)$$

The sign of the transverse component e_T of the eccentricity vector depends on the sign of $\mathbf{e} \cdot \hat{\mathbf{i}}_p$, whereas in Battin's notation, where the fundamental ellipse eccentricity is defined in terms of absolute value, the transverse component $|e_F| \tan(\Delta\omega)$ is positive if the argument of the periastris increases. Expressing $|e_F| = \sigma(e_F)e_F$ and noting that $\sigma(e_F) = -\sigma(r_2 - r_1)$, it is possible to write

$$\sigma(r_2 - r_1)|e_F| = -\sigma(e_F)[\sigma(e_F)e_F] = -e_F$$

so that Eq. (5) can be rewritten as

$$p(\Delta\omega) = p_F - e_F \tan(\Delta\omega) \frac{r_1 r_2}{c} \sin(\Delta\theta) \quad (9)$$

where the component of the eccentricity vector of the fundamental ellipse along $\hat{\mathbf{i}}_c$, namely, e_F as given by Eq. (6), is used with its sign.

To derive a general parametrization of the orbits that is also valid when the minimum eccentricity is zero, the transverse component $e_T = e_F \tan \Delta\omega$ that appears in Eq. (7) will be used in the sequel as the free parameter, rather than the periastris rotation $\Delta\omega$. The orbit parameter it thus expressed as

$$p(e_T) = p_F - e_T \frac{r_1 r_2}{c} \sin(\Delta\theta) \quad (10)$$

where the last expression does not feature the highly nonlinear sign function and accommodates for circular fundamental ellipses. Finally, the semimajor axis is given by

$$a(e_T) = \frac{p(e_T)}{1 - e_F^2 - e_T^2}$$

Letting ω_c be the angle between the reference direction $\hat{\mathbf{r}}_1$ and the chord $\hat{\mathbf{i}}_c$ (Fig. 2), it is possible to express $\hat{\mathbf{i}}_c$ as $(\cos \omega_c, \sin \omega_c, 0)^T$ and $\hat{\mathbf{i}}_p = (-\sin \omega_c, \cos \omega_c, 0)^T$. The coordinates of the generic eccentricity vector $\mathbf{e}(e_T)$ are thus given by

$$\begin{aligned} \mathbf{e}(e_T) &= (e \cos \omega, e \sin \omega, 0)^T \\ &= (e_F \cos \omega_c - e_T \sin \omega_c, e_F \sin \omega_c + e_T \cos \omega_c, 0)^T \end{aligned}$$

The argument of the periastris of the generic orbit is determined from the equation

$$\omega(e_T) = \tan^{-1}(e_F \sin \omega_c + e_T \cos \omega_c, e_F \cos \omega_c - e_T \sin \omega_c) \quad (11)$$

where $\tan^{-1}(y, x)$ is the four-quadrant inverse tangent function. The true anomaly of P_1 and P_2 will thus be given by $\nu_1 = -\omega$ and $\nu_2 = \Delta\theta - \omega$. It is now possible to evaluate the time-of-flight between P_1 and P_2 as a function of e_T by use of Kepler's time equation. Standard eccentric, parabolic, or hyperbolic anomaly will be used, depending on whether the eccentricity $e = \|\mathbf{e}\|$ is less than, equal to, or greater than one.

B. Limits to the Variation of e_T

Before discussing the parametrization of the transfer time in terms of the transverse eccentricity component e_T , it must be emphasized that this parameter cannot be varied arbitrarily. As long as the orbits remain elliptical, the transfer from P_1 to P_2 is accomplished in a finite time along either the shorter or the longer arc on the considered elliptical orbit. Letting $e_p = (1 - e_F^2)^{1/2}$ be the value of e_T that makes the orbit parabolic, all the values of the transverse eccentricity component $|e_T| < e_p$ correspond to admissible elliptical transfer orbits.

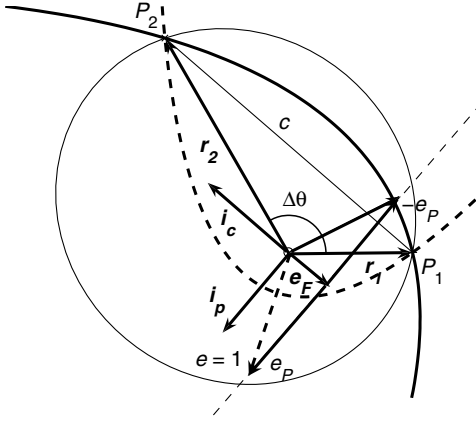


Fig. 3 Admissible parabolic orbit (continuous line) and diverging one (dashed line).

Two parabolic orbits are present for $e_T = \pm e_p$ (Fig. 3), but only one of them is admissible (that is, allows for a transfer from P_1 to P_2). If the direction opposite to the periapsis of the parabolic orbit lies inside the transfer angle $\Delta\theta$ (that is, $-e$ lies between r_1 and r_2), the orbiting body does not reach P_2 anymore after leaving P_1 , and the transfer time along the parabolic orbit becomes infinite (dashed line in Fig. 3). It can be shown on geometrical grounds that, whichever the value and the sign of e_F , a parabolic escape trajectory from P_1 is always accomplished for $e_T = +e_p$, so that a first limit on the variation of the transverse eccentricity component is $e_T < e_p$. Any further rotation of the eccentricity vector related to an increase of e_T beyond e_p will provide hyperbolic orbits that depart from P_1 without allowing for the desired transfer from P_1 to P_2 .

When $e_T = -e_p$, the vector e lies inside of the angle between r_1 and r_2 , so that P_1 is connected to P_2 by an arc of an admissible parabolic trajectory (continuous line in Fig. 3). Further increments of the eccentricity in the negative transverse direction will allow for faster transfers along arcs of hyperbolic trajectories.

The requirement $e_T < e_p$ is the only limitation on the value of e_T as far as the transfer angle remains smaller than or equal to π . A negative transverse component of the eccentricity vector is virtually unlimited for $\Delta\theta \leq \pi$. The “infinite eccentricity” orbit for $e_T \rightarrow \infty$ corresponds to a hyperbolic orbit, where the body travels at infinite velocity along a rectilinear path between P_1 and P_2 .

Conversely, if $\Delta\theta > \pi$, there is a limitation on the maximum admissible negative value for e_T . From the geometry of hyperbolic orbits described by Eq. (2), $r \rightarrow \infty$ when $\cos \nu \rightarrow (-1/e)$, so that the total angular travel along the hyperbolic orbit is $\Delta\nu = 2\cos^{-1}(-1/e)$. To be feasible, the angular travel required by Lambert’s problem $\Delta\theta$ must remain smaller than $\Delta\nu$. This condition holds for a prescribed value of $\Delta\theta$ if the orbit eccentricity satisfies the inequality

$$e < e_{\max} = -\frac{1}{\cos(\Delta\theta/2)}$$

where e_{\max} is a positive number larger than one, if $\Delta\theta > \pi$. The transverse component is thus limited in the negative direction of the transverse axis, where hyperbolic transfers are admissible, by the condition $e_T > -e_H = -(e_{\max}^2 - e_F^2)^{1/2}$.

Note that if e_T approaches its limit and $e \rightarrow -1/\cos(\Delta\theta/2)$, the transfer from P_1 to P_2 is accomplished by “flying the asymptotes” that connect P_1 to F and F to P_2 at infinite velocity in zero time. This solution is infeasible, which explains the requirement for a strict inequality.

Summarizing, the transverse component of the eccentricity vector must lie between $-e_H$ and e_p . With some simple trigonometric manipulations, based on the theorem of cosines, it is possible to state the bounds on e_T in the form

$$-\frac{r_1 + r_2}{c} \sqrt{\frac{1 - \cos \Delta\theta}{1 + \cos \Delta\theta}} < e_T < \frac{1}{c} \sqrt{2r_1 r_2 (1 - \cos \Delta\theta)} \quad (12)$$

Table 1 Transfer configurations

	$r_2 \geq r_1$ ($\rho \geq 1$)	$r_2 < r_1$ ($\rho < 1$)
$\Delta\theta \leq \pi$	Case 1 $e_F < 0$ $e_T < e_p$	Case 2 $e_F > 0$ $e_T < e_p$
$\Delta\theta > \pi$	Case 3 $e_F < 0$ $-e_H < e_T < e_p$	Case 4 $e_F > 0$ $-e_H < e_T < e_p$

where the inequality on the left-hand side needs to be enforced only if $\Delta\theta > \pi$.

C. Orbit Parametrization and Transfer Times

It is now possible to analyze the variation of orbit parameters and transfer time between P_1 and P_2 for the admissible range of variation of the transverse component of the eccentricity vector. From the qualitative point of view, a combination of four different cases is possible, depending on whether or not $r_2/r_1 > 1$ and $\Delta\theta > \pi$. The cases are listed in Table 1.

In what follows, all the distances are scaled with respect to r_1 , whereas time is scaled with respect to the reference time $t_{\text{ref}} = 1/n_{\text{ref}}$, where $n_{\text{ref}} = \sqrt{\mu/r_1^3}$ is the orbit rate of a circular orbit of radius r_1 . This is equivalent to assuming $r_1 = 1$ and $\mu = 1$. In this way, all the geometrical information is expressed in terms of two parameters only, namely, the ratio of the radii $\rho = r_2/r_1$ and the transfer angle $\Delta\theta$. By letting $\tau_{12} = t_{12}n_{\text{ref}}$, the nondimensional transfer time becomes

$$\tau_{12} = f(\rho, \Delta\theta; e_T) \quad (13)$$

where the transverse eccentricity component e_T is the only unknown. An interesting aspect of this nondimensional parametrization is that Lambert’s problem is rephrased as a root-finding problem for the equation

$$f(\rho, \Delta\theta; e_T) - \left[t_{12}^S \sqrt{\frac{\mu}{r_1^3}} \right] = 0 \quad (14)$$

where all the dimensional information on the problem is in the coefficient between square brackets. Before discussing the solution of the problem, which will be covered in the next section, some relevant properties of the parametrization in terms of e_T will be outlined, which strongly support the validity of the approach proposed.

Figure 4 shows the structure of admissible orbits for a transfer angle $\Delta\theta = 2\pi/3$ and a radius ratio $\rho = 2$, which is an example of transfers for a case 1 configuration. Elliptical admissible trajectories are represented by continuous lines, the thickest one showing the fundamental ellipse, whereas hyperbolic trajectories through P_1 and P_2 are indicated by dashed lines. The diverging parabolic trajectory,

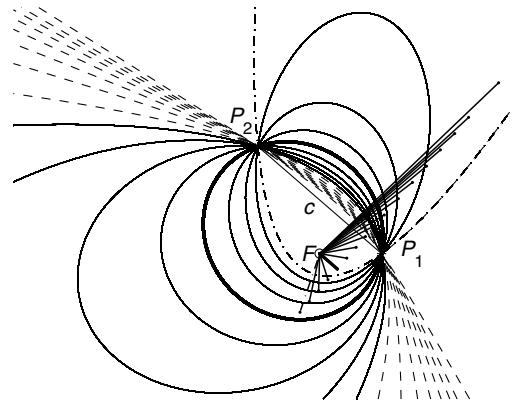


Fig. 4 Admissible orbits for case 1 ($\Delta\theta = 120$ deg, $r_2/r_1 = 2$).

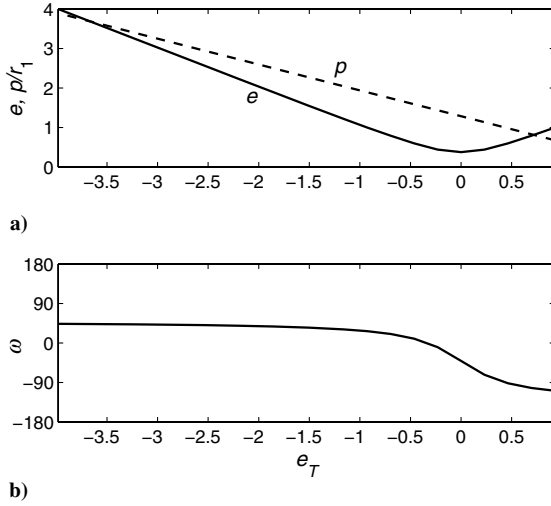


Fig. 5 Orbit parameters as a function of e_T ($\Delta\theta = 120$ deg, $r_2/r_1 = 2$).

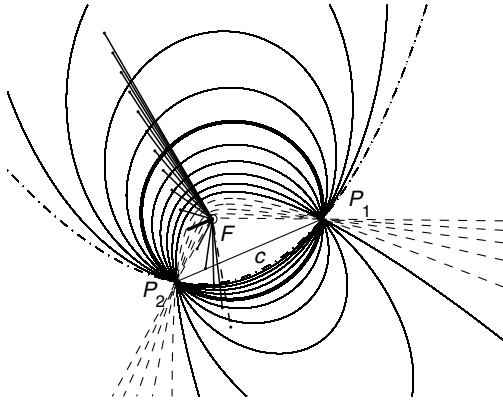


Fig. 6 Admissible orbits for case 4 ($\Delta\theta = 240$ deg, $r_2/r_1 = 0.66$).

which limits the admissible set for positive values of the transverse eccentricity component, is also plotted (dash-dot line). Finally, the segments based on the focus F represent the eccentricity vectors of the orbits, which terminate on a straight line perpendicular to the chord c . The corresponding variation of orbit parameters is reported in Fig. 5. As expected from Eq. (10), the variation of the parameter p with e_T is linear (dashed line in Fig. 5a), the slope being negative for $\Delta\theta \leq \pi$.

When $\Delta\theta > \pi$, the geometry of admissible orbits changes. Figure 6 reports an example for a transfer angle $\Delta\theta = 4\pi/3$ and a radius ratio $\rho = 0.66$ (case 4 configuration), where elliptical and hyperbolic orbits are represented as before by continuous and dashed lines, respectively. Figure 7 shows the variation of orbit parameters, where the slope of the dashed line representing p is positive. In this respect, it is possible to provide a new interpretation on the limit for e_T , inasmuch as for $e_T = -e_H$ it is $p = 0$, so that e_H can be derived directly from Eq. (10) as $e_H = -cp_F/(r_1 r_2 \sin \Delta\theta)$.

The variation of ω , evaluated according to Eq. (11), presents a discontinuity when $r_2 < r_1$ (Fig. 7b). This discontinuity does not affect the variation of all the other parameters of interest, including transfer time. More important, the chosen parametrization preserves the direction of the slope of the function τ_{12} for all the four possible cases considered. Four representative examples of nondimensional transfer times as a function of e_T are reported in Fig. 8, showing how the variation of τ_{12} is always monotonically increasing between the minimum admissible negative value of e_T (possibly $-\infty$ for $\Delta\theta \leq \pi$) and the maximum positive one (always equal to e_p). A fifth case is added for $\Delta\theta = \pi$ (thin continuous line with dots in Fig. 8) to demonstrate that no singularity is present, and a smooth monotonic variation of τ_{12} is also obtained in this critical case. This fact paves

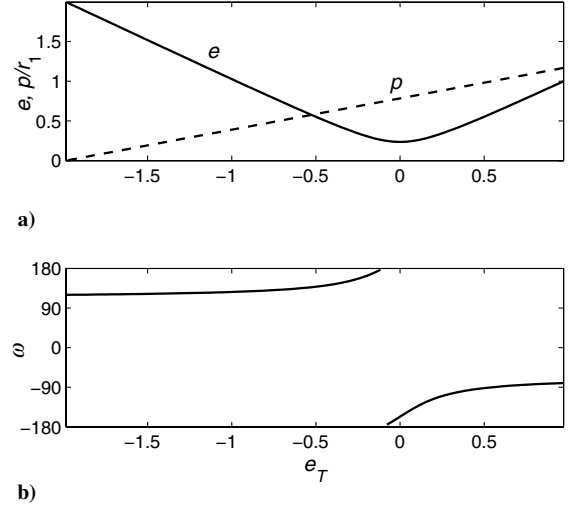


Fig. 7 Orbit parameters as a function of e_T ($\Delta\theta = 240$ deg, $r_2/r_1 = 0.66$).

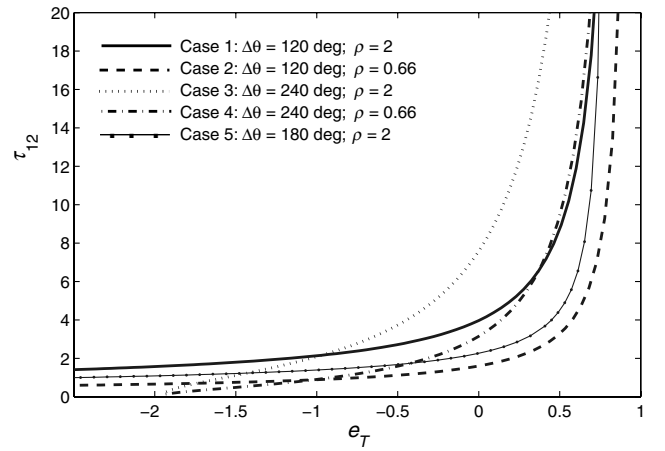


Fig. 8 Transfer time between P_1 and P_2 as a function of e_T .

the way for a uniform numerical approach to the solution of Lambert's problem, inasmuch as the algorithm will not depend on the particular case considered.

IV. Solution of Lambert's Problem by Newton Iterations

A. Coordinate Transformation

Given a monotonic variation of the target parameter (that is, the transfer time in nondimensional form τ_{12}) as a function of the design parameter (the transverse eccentricity component e_T), a numerical algorithm needs to be used for searching the value of e_T in the admissible range that corresponds to the orbit with the specified transfer time τ_{12}^S .

Newton's method provides a fast convergence rate, at the reasonable price of two evaluations of the transfer time per iteration, to numerically determine the local derivative. Unfortunately, in the case of a poor initial guess, it is easy to leave the admissible domain and cause the algorithm to become stuck in intervals of the e_T axis, where the transfer time is no longer defined. To provide a fast convergence rate without running this risk, a coordinate transformation $x = \xi(e_T)$ is devised, which allows for a fail-proof application of the Newton-Raphson algorithm.

The coordinate transformation law is chosen in such a way that the transformed variable diverges toward plus or minus infinity if the corresponding value of the transverse eccentricity component approaches one of its limiting values. When $-e_H < e_T < e_p$, it is

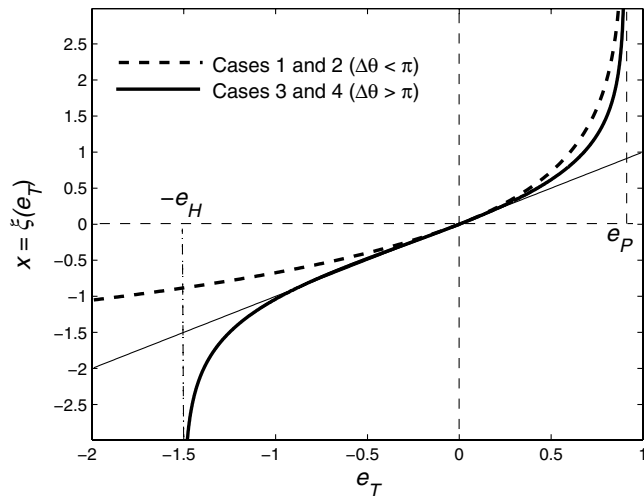


Fig. 9 Coordinate transformations for $\Delta\theta \leq \pi$ (dashed line) and $\Delta\theta > \pi$ (continuous line).

$$x = \frac{e_P e_H}{e_P + e_H} \log \left(\frac{e_P e_T + e_H}{e_H e_P - e_T} \right) \quad (15)$$

Note that when $e_T \rightarrow e_P$, the argument of the log function approaches $+\infty$ and x grows unboundedly, whereas, if $e_T \rightarrow -e_H$, the argument of the logarithm approaches zero and x diverges toward $-\infty$ (continuous line in Fig. 9). At the same time, when $e_T = 0$, it is $x = 0$ and $dx/de_T = 1$, which means that, in the neighborhood of the fundamental ellipse, the variation of x matches that of e_T up to the first order. Letting

$$X = \exp \left[\left(\frac{1}{e_H} + \frac{1}{e_P} \right) x \right]$$

the inverse transformation is given by

$$e_T = e_P e_H \frac{X - 1}{e_P + e_H X} \quad (16)$$

When $\Delta\theta \leq \pi$, there is no limit for negative increments of the transverse component. In this case, the coordinate transformation law and its inverse are expressed as

$$x = -e_P \log \left(1 - \frac{e_T}{e_P} \right) \quad (17)$$

$$e_T = e_P \left[1 - \exp \left(-\frac{x}{e_P} \right) \right] \quad (18)$$

(dashed line in Fig. 9). If the logarithm of the nondimensional transfer time τ_{12} is expressed as a function of the transformed variable x , plots like those reported in Fig. 10 are obtained, which feature a quasi-linear behavior outside of a relatively small neighborhood of the fundamental ellipse case $x = 0$. This qualitative behavior is always present unless degenerate cases are dealt with, when $\Delta\theta$ is very close to either zero or 2π , and quasi-rectilinear transfers along almost radial orbits are considered.

B. Solution of Lambert's Problem for Direct Transfers

Indicating the direct and inverse coordinate transformation laws with $x = \xi(e_T)$ and $e_T = \xi^{-1}(x)$, respectively, while $\Delta\theta$ and $\rho = r_2/r_1$ describe the problem geometry, it is possible to rephrase Lambert's problem as a root-finding problem in the form

$$g(x) - y^S = 0$$

where $y^S = \log(\tau_{12}^S)$ and $g(x) = \log\{f[\Delta\theta, \rho; \xi^{-1}(x)]\}$. The solution is obtained by means of a Newton–Raphson iterative scheme.

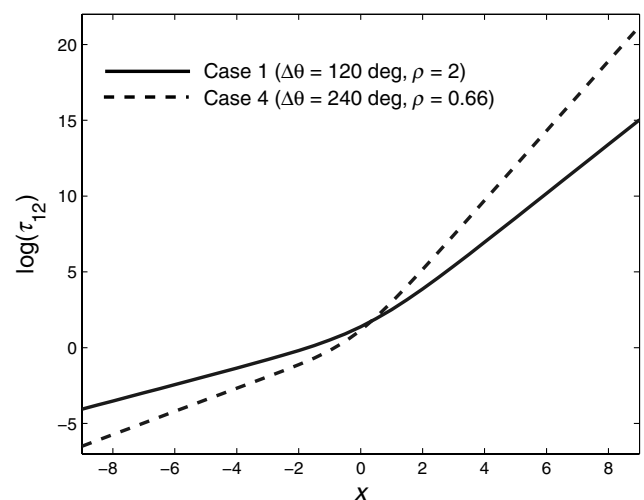


Fig. 10 Nondimensional transfer time for case 1 (continuous line) and case 4 (dashed line) in terms of transformed coordinates.

A flowchart of the numerical algorithm is reported in Fig. 11. After the input of problem data, the type of coordinate transformation for the unknown variable is selected, depending on whether $\Delta\theta > \pi$ [cases 3 and 4, Eq. (16)] or not [cases 1 and 2, Eq. (18)]. The initial guess for the unknown parameter is $x_0 = 0$ for both cases, that is, the nondimensional transfer time along an arc of the fundamental ellipse is evaluated during the first iteration. If the result is not sufficiently close to the prescribed value y^S , the unknown variable is incremented, and the derivative of $y = \log \tau_{12}$ with respect to x is evaluated numerically by forward differences. The k th iteration is thus given by

$$x_{k+1} = x_k + \left(\frac{dy}{dx} \bigg|_{x_k} \right)^{-1} [y^S - g(x_k)]$$

The counter k is increased by one unit and the iteration procedure is repeated until either convergence criteria on error and step-size are satisfied or the maximum number of iterations k_{\max} is exceeded. In the first case, the orbit parameters for the desired transfer are evaluated.

Very strict tolerances are adopted for the two convergence criteria $|y^S - g(x_k)| < \varepsilon_y$ and $|x_{k+1} - x_k| < \varepsilon_x$, with $\varepsilon_y = 10^{-12}$ and $\varepsilon_x = 10^{-6}$. A relaxation on ε_y is allowed when τ_{12}^S approaches zero. This is accomplished by letting $\varepsilon_y = 10^{-12}/\tau_{12}^S$ if $\tau_{12}^S < 1$. Use of a logarithmic coordinate transformation results in an absolute precision in the determination of the nondimensional transfer time τ_{12} still equal to 10^{-12} . Conversely, when τ_{12}^S becomes large, the logarithmic transformation partially spoils the absolute accuracy of the solution, but the number of significant figures correctly evaluated is still equal to 12.

A similar relaxation technique is used when $\Delta\theta$ is close to either zero or 2π (that is, $\min(|\Delta\theta|, |2\pi - \Delta\theta|) < 0.02$ rad and the admissible range for e_T becomes narrower). Note that all the aforementioned cases correspond to situations of scarce practical interest (if any), where the solution orbit is either close to the limits of the admissible variation with very large or very small desired transfer times, or the set of admissible orbits is in a narrow neighborhood of radial trajectories.

The algorithm was tested on a series of 1000 randomly generated sample problems, with values of $\Delta\theta$ between zero and 2π rad, ρ ranging from 0.01 to 10, and the desired nondimensional transfer time varying between 0.01 and 20. By using $x_0 = 0$ as the initial guess (that is, $e_T = 0$), convergence is achieved in a number of iterations between two and seven, with 99.5% of the cases reaching convergence in five iterations or less. Comparison with the Battin–Vaughan method [8] on the same set of sample problems shows that the novel technique is not as efficient as the classic one, in spite of the fact that Battin's algorithm usually requires a slightly higher number

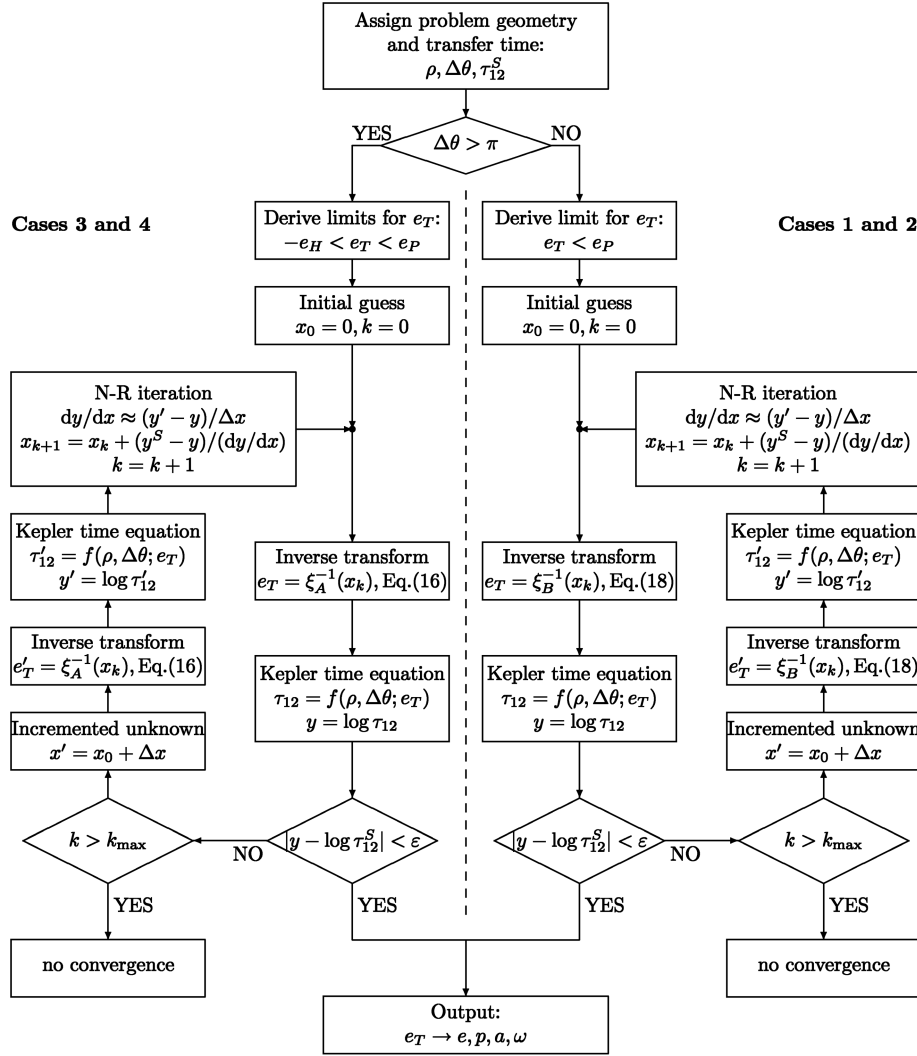


Fig. 11 Flowchart of the Newton–Raphson (N-R) iteration scheme.

of iterations to reach the same accuracy, less than 90% of the cases being solved with five or less iterations.

The higher computation time is due to the fact that the determination of the transfer time for a given value of the design parameter e_T requires the evaluation of Kepler's equation twice. Moreover, the initial guess is not optimized, and some improvements can be expected by a more accurate numerical implementation. Nonetheless, the difference in terms of CPU time is usually negligible and never exceeds 30% in the worst cases. Among the advantages of the new method, the physical insight is deeper, because the parametrization works directly on the set of admissible orbits, and the mathematical implementation is significantly easier, with no hypergeometric functions being involved. The coordinate transformations [Eqs. (15–18)] are required only for speeding up convergence and preventing divergence toward inadmissible values of the transverse eccentricity component. As a result, a small penalty in terms of computational efficiency appears to be a modest price to pay for a significantly less demanding algorithm, the implementation of which requires only a basic knowledge of the fundamental principles of astrodynamics.

V. Conclusions

A classical result from astrodynamics was used to derive a novel parametrization of orbits passing through two points in space in the framework of the two-body problem. In this respect, the eccentricity vector of the generic orbit passing through the two points is decomposed into a constant component parallel to the chord connecting the two points and a variable transverse component in the

direction perpendicular to it, on the orbit plane. The limits of variation of the transverse component were identified and a coordinate transformation was devised that allows for a parametrization between $-\infty$ and $+\infty$.

In this way, it is possible to employ a simple Newton–Raphson iterative scheme to solve Lambert's problem, that is, the determination of the orbit connecting the two points with a specified transfer time. The method was applied to the direct transfer case, that is, the angular travel of the orbiting body is smaller than a complete revolution around the occupied focus. As a major advantage, together with the absence of singularities and in spite of suboptimal numerical performance, the proposed parametrization results in a very simple solution algorithm.

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